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OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

by

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subject to $d\xi_t = u_t(\xi)dt + dw_t$, $\xi_0 = x$; $|u| \le 1$, w_t Wiener, with all bounded by unity and measurable functionals on the past of the

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$$u_t(\xi) = -1,\quad \xi_t > b$$
$$= 0,\quad |\xi_t| \leq b$$
$$= 1,\quad \xi_t < -b.$$

The cutoff point b and the performance rate of the optimal law u^* are simultaneously determined in terms of the function $\phi(\cdot)$ through a simple system of integrotranscendental equations.

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OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

Václav E. Beneš and Ioannis Karatzas

ABSTRACT

In the present paper we consider the following stochastic control problem: minimize the average expected total cost

$$J(x,u) = \liminf_{T\to\infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|]dt$$

subject to $d\xi_t = u_t(\xi)dt + dw_t$, $\xi_0 = x$; $|u| \le 1$, w_t Wiener, with all bounded by unity and measurable functionals on the past of the state process $\{\xi_s;\ s \le t\}$ admissible as controls. It is proved that under very mild conditions on the running cost function $\phi(\cdot)$ the optimal law is of the form

$$\begin{aligned} u_t^*(\xi) &= -1,\ \xi_t > b \\ &= \ \ 0,\ |\xi_t| \le b \\ &= \ \ 1,\ \xi_t < -b. \end{aligned}$$

The cutoff point b and the performance rate of the optimal law u^* are simultaneously determined in terms of the function $\phi(\cdot)$ through a simple system of integrotranscendental equations.

OPTIMAL STATIONARY LINEAR CONTROL OF THE WIENER PROCESS

1. INTRODUCTION

In this paper we consider the problem of stationary control of the stochastic differential equation, $d\xi_t = u_t(\xi)dt + dw_t;\ \xi_0 = x$, where $\cancel{\{w_t\}}$ $\cancel{= \{w_t;\ t \geq 0\}}$ is a Wiener process on an underlying probability space, $\cancel{(\Omega, \mathcal{F}, P)}$.

(w sub t)

Two kinds of cost are involved in this problem. First, one
Phi (zeta sub t) zeta sub t
pays $\cancel{\phi(\xi_t)}$, per unit time for being in the wrong state $\cancel{\xi_t}$, where $\phi(\cdot)$ is a suitable cost function to be described later; secondly, one pays $\cancel{|u_t|}$ per unit time for using the control
u sub T
law $\cancel{u_t}$. The control problem is to choose a law $u_t(\xi)$ as a non-anticipative functional of the solution process (ξ_t) with values in the bounded interval $[-1,1]$, so as to minimize the average expected total cost.

It is proved that the optimal law can be explicitly described and its performance characterized in terms of the cost function $\phi(\cdot)$. The method consists in first restricting attention to an important subclass of admissible control laws, namely those giving rise to an ergodic solution process (ξ_t). A process is said to be ergodic if it admits a unique invariant distribution. The optimal law u^* in this subclass can be obtained by using a dynamic programming approach, similar to that of Wonham [11]; it turns out that u^* is of the form

$$u_t^*(\xi) = -\operatorname{sgn} \xi_t, \quad |\xi_t| > b \tag{1.1}$$
$$= 0, \quad |\xi_t| \leq b$$

where b is a positive constant that can be characterized in terms of the function $\phi(\cdot)$. Secondly, the law u^* is proved optimal against any possible nonanticipative law u whatsoever.

The result (1.1) is the natural and expected one; it says that the best policy is to push ξ_t with full force in the negative direction if it is too positive and in the negative direction if it is too negative, while refraining from any action if ξ_t is inside a "dead-zone" $[-b,b]$. The appearance of the latter is a consequence of the running cost $|u|$ on the control, of the fact that the control is "expensive". Were such a cost absent, it is fairly obvious - ane easily probable by using the methods of the present paper - that the optimal policy would be described by the "bang-bang" law: $-\mathrm{sgn}\, \xi_t$.

Among previous works on the topic of stationary control of systems driven by a Wiener process we cite those of Wonham [11] and Kushner [8]. The scope of both was severely restricted, however, in that they allowed only those laws that generate an ergodic solutio process (actually, only a subclass of these was considered).

2. FORMULATION

Consider the space $\Omega = C_{[0,T]}$ of real-valued, continuous functions on $[0,T]$, for some $T > 0$. Let (ξ_t) denote the family of evaluation functionals on $C_{[0,T]}$ and $\mathscr{F}_t, 0 \leq t \leq T$ the σ-field of subsets of $C_{[0,T]}$ generated by $\{\xi_s;\ s \leq t\}$.

Consider also the σ-field $\mathscr{M}$ of subsets M of $[0,T] \times C_{[0,T]}$ having the property that, for any $t \in [0,T]$, M_t belongs to $\mathscr{F}_t$ and that each ξ-section M_ξ of M, $\xi \in C_{[0,T]}$, is Lebesgue measurabl

A function g defined on $[0,T] \times C[0,T]$ is $\mathscr{M}$-measurable if and only if $g(t,\cdot)$ is $\mathscr{M}_t$-measurable, for each t, and $g(\cdot,\xi)$ is Lebesgue measurable, for each ξ.

Definition 2.1: Let the control measure space be the interval $[-1,1]$ with its Borel sets. An admissible nonanticipative control law u is a measurable function $u\colon ([0,T] \times C_{[0,T]}, \mathscr{M}) \to [-1,1]$. The class of all such control laws is denoted by $\mathscr{U}$.

For any control law $u \in \mathscr{U}$ and any $x \in \mathbb{R}$, a weak solution (ξ_t) to the stochastic differential equation

$$d\xi_t = u_t(\xi)dt + dw_t; \quad 0 \leq t \leq T \tag{2.1}$$

$$\xi_0 = x \tag{2.2}$$

is constructed as follows: one starts with the probability space $(\Omega, \mathscr{F}_T, P)$, where P is Wiener measure on $\Omega = C_{[0,T]}$. Corresponding to each law $u \in \mathscr{U}$ and each initial position $x \in \mathbb{R}$, the new measure

$$P_x^u(d\omega) = \exp\left[\int_0^T u_t(\xi)dw_t - \frac{1}{2}\int_0^T u_t^2(\xi)dt\right]\cdot P(d\omega) \tag{2.3}$$

is constructed on $(\Omega, \mathscr{F}_T)$, where (ξ_t) is the process defined by $\xi_t = x + w_t$; $0 \leq t \leq T$. According to Girsanov [5], P_x^u is a probability measure on $(\Omega, \mathscr{F}_T)$ and the process

$$\tilde{w}_t \triangleq w_t - \int_0^t u_s(\xi)ds = \xi_t - x - \int_0^t u_s(\xi)ds \tag{2.4}$$

is a Wiener process on $(\Omega, \mathcal{F}_T, P_x^u)$. Equation (2.4) is an equival[ent] way of saying that the process (ξ_t), $\xi_t = x + w_t$; $0 \leq t \leq T$ satisfies the stochastic differential equation

$$d\xi_t = u_t(\xi)dt + d\tilde{w}_t; \quad 0 \leq t \leq T \tag{2.1)'$$

$$\xi_0 = x \tag{2.2}$$

on $(\Omega, \mathcal{F}_T, P_x^u)$. All processes involved here are adapted to the underlying family $(\mathcal{F}_t)$ of sub-σ-fields of $\mathcal{F}_T$. The process (ξ_t) is called a weak solution of (2.1)'-(2.2) because by construction $\sigma(\tilde{w}_s;\ s \leq t) \subseteq \sigma(\xi_s;\ s \leq t)$, though not necessarily the other way around. Such a solution is known to be unique in the sense of the probability law; see Liptser and Shiryayev [9].

Now consider a function $\phi: \mathbb{R} \to \mathbb{R}^+$ which is even, convex piecewise $C^{(2)}$, monotonically increasing to infinity on $x > 0$, and satisfying an exponential growth condition:

$$\phi(x) = O(e^{\alpha|x|}) \quad \text{as} \quad |x| \to \infty, \text{ some } \ 0 < \alpha < 2. \tag{2.5}$$

The optimal control problem can now be formulated as follows: choose a law $u \in \mathcal{U}$ for which the limit

$$J(x,u^*) = \lim_{T\to\infty} \frac{1}{T} E_x^{u^*}\int_0^T (\phi(\xi_t) + |u_t^*(\xi)|)dt$$

exists for all $x \in \mathbb{R}$, and which minimizes the average expected total cost rate

$$J(x,u) = \liminf_{T\to\infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|]dt \tag{2.6}$$

of starting at place x and using control law u, for all $(x,u) \in [\mathbb{R} \times \mathscr{U}]$. E_x^u denotes expectation with respect to the probability measure P_x^u introduced in (2.3).

3. SUMMARY

In Section 4 we briefly study the important subclass of feedback (Markov) admissible control laws. It is pointed out (and in the special case of time-homogeneous feedback laws, proved) that for such controls the stochastic differential equation (2.1)-(2.2) o the system can be solved in the strong sense.

In Section 5 we consider a subclass of time-homogeneous feedback laws that give rise to an ergodic solution process. Asymptotic properties of those processes, such as existence of a unique invaria measure, laws of large numbers and ergodicity of their distributions are discussed.

The optimal law u^* in the abovementioned subclass is discerned in Section 6 and it is proved that u^* is of the form (1.1). Both the cutoff point b and the asymptotic performances λ rate of u^* are characterized in terms of the cost function $\phi(\cdot)$, through the system of integrotranscendental equations (6.3), (6.4). The method proceeds by constructing a solution to the "asymptotic" version of the Bellman equation of dynamic programming (6.2).

Finally, the asymptotic performance of the law u^* is compared against that of any admissible nonanticipative control u in $\mathscr{U}$. The result, proved in Section 7, is that u^* is actually optimal in the (largest possible) class $\mathscr{U}$. The idea employed here is to first compare the performance of the control laws over finite time intervals $[0,T]$ and then pass to the limit as $T \to$

4. MARKOV LAWS AND STRONG SOLUTIONS

Definition 4.1. Suppose there exists a measurable function $\gamma: \mathbb{R} \times [0,T] \to [-1,1]$ such that the nonanticipative law $u \in \mathscr{U}$ can be represented in the form

$$(4.1) \qquad u_t(\xi) = \gamma(\xi_t, t), \text{ any } \xi \in C_{[0,T]}, \quad 0 \leq t \leq T.$$

Then u is called an admissible Markov law. The class of all such laws will henceforth be denoted by $\mathscr{A}$; obviously $\mathscr{A} \subseteq \mathscr{U}$.

For laws in $\mathscr{A}$ the stochastic differential equation

$$(4.2) \qquad d\xi_t = \gamma(\xi_t, t)dt + dw_t, \quad \xi_0 = x$$

is known to possess a pathwise unique, strong nonanticipative solution, in the sense that the solution is adapted to the Wiener process: $\sigma(\xi_s; s \leq t) \subseteq \sigma(w_s; s \leq t)$, $0 \leq t \leq T$; see Zvonkin [12]

Definition 4.2. Consider the subclass of $\mathscr{A}$ consisting of those admissible nonanticipative laws u for which there exists a measurable function $a: \mathbb{R} \to [-1,1]$, such that

$$(4.3) \qquad u_t(\xi) = a(\xi_t), \text{ any } \xi \in C_{[0,T]}, \; 0 \leq t \leq T.$$

Such laws u are called <u>admissible time-homogeneous Markov laws</u> and their class is denoted by $\mathscr{A}$.

For laws in $\mathscr{A}$ one can easily construct the (pathwise unique) strong solution to the stochastic differential equation

$$
(4.4)\qquad \begin{aligned} d\xi_t &= a(\xi_t)dt + dw_t, \quad 0 \le t \le T \\ \xi_0 &= x. \end{aligned}
$$

Indeed, consider the function

$$
(4.5)\qquad \beta(x) = \int_0^x \exp\{-2\int_0^y a(z)dz\}dy; \; x \in \mathbb{R}
$$

which is continuous, strictly increasing and satisfies the equation $\beta'' + 2a\beta' = 0$. The function

$$
(4.6)\qquad \sigma(x) = \beta'(\beta^{-1}(x)); \quad x \in \mathbb{R}
$$

is Lipschitz continuous, as can be checked by simple calculus. Therefore the stochastic differential equation

$$
(4.7)\qquad d\zeta_t = \sigma(\zeta_t)dw_t; \; 0 \le t \le T
$$

$$
(4.8)\qquad \zeta_0 = \beta(x)
$$

has for any $x \in \mathbb{R}$ a pathwise unique solution (ζ_t) on the probability space $(\Omega, \mathscr{F}_T, P)$, strong in the sense that

$\sigma(\zeta_s;\ s \leq t) \subseteq \sigma(w_s;\ s \leq t)$, any $0 \leq t \leq T$, according to Itô's classical theory; see for instance Gihman and Skorohod [4]. Denote by $\{\Omega, \mathscr{F}_T, \mathscr{F}_t, \zeta_t, P^u_{\beta(x)}\}$ the corresponding time-homogeneous Markov process

The process

(4.9) $$\xi_t = \beta^{-1}(\zeta_t)$$

is now well defined, and an application of Itô's rule gives

$$d\xi_t = \frac{1}{\beta'(\beta^{-1}(\zeta_t))}\, d\zeta_t - \frac{1}{2}\, \frac{\beta''(\beta^{-1}(\zeta_t))}{(\beta'(\beta^{-1}(\zeta_t)))^3}\ \sigma^2(\zeta_t)dt$$

$$= a(\xi_t)dt + dw_t.$$

So (ξ_t) satisfies both the equation and the initial condition in (4.4) and because it is a bijection of (ζ_t) pointwise in time:

$$\sigma\{\xi_s;\ s \leq t\} = \sigma\{\zeta_s;\ s \leq t\} \subseteq \sigma\{w_s;\ s \leq t\}, \quad 0 \leq t \leq T$$

i.e. (ξ_t) is a strong solution to (4.4). The corresponding time-homogeneous Markov process is denoted by $\{\Omega, \mathscr{F}_T, \mathscr{F}_t, \xi_t, P^u_x\}$.

5. SOME ERGODIC THEOREMS

Introduce the function $G(x) \triangleq \int_{-\infty}^{x} \frac{dz}{\sigma^2(z)}$, $\sigma(\cdot)$ as in (4.6), and consider the subclass $\mathscr{U}'$ of $\mathscr{U}$, consisting of those laws

u, $u_t(\xi) = a(\xi_t)$ for which

$$(5.1) \qquad G(\infty) = \int_{-\infty}^{\infty} \frac{dz}{\sigma^2(z)} = \int_{-\infty}^{\infty} \exp\{2 \int_0^y a(z)dz\}dy < \infty;$$

recall also the processes $(\xi_t),(\zeta_t)$ of the preceding section, corresponding to this law. According to Gihman and Skorohod [4; §18], the probability distribution $\frac{G(\cdot)}{G(\infty)}$ is ergodic for the Markov process $\{\Omega, \mathscr{F}_T, \mathscr{F}_t, \zeta_t, P_z\}$ in the sense that the following are true:

Fact 1. Positive Recurrence: The stopping times $\tau_{zy} = \inf\{t: \zeta_t = y\}$ are well defined and a.s. finite for any $z, y \in \mathbb{R}$; besides,

$$(5.2) \qquad E_z^u(\tau_{zy}) < G(\infty)(2+|z-y|)|z-y|.$$

Fact 2. Invariance of the Probability Distribution Function $G(\cdot)/G$

For any $0 \leq t \leq T$,

$$(5.3) \qquad \int_{-\infty}^{\infty} P_z^u\{\zeta_t \leq y\}dG(z) = G(y), \quad y \in \mathbb{R}.$$

Fact 3. Law of Large Numbers: For any Borel function $f(\cdot)$ such that $\int_{-\infty}^{\infty} |f(y)|dG(y) < \infty$, we have

$$(5.4) \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\zeta_t)dt = \frac{1}{G(\infty)} \int_{-\infty}^{\infty} f(y)dG(y);$$

a.s. (P_z^u) and $L^1(E_z^u)$, any $z = \zeta_0 \in \mathbb{R}$.

Fact 4. Ergodicity of the Distributions: For any function $f(\cdot)$ as above,

$$\lim_{t\to\infty} E_z^u f(\zeta_t) = \frac{1}{G(\infty)} \int_{-\infty}^{\infty} f(y)dG(y), \text{ any } \quad z = \zeta_0. \tag{5.5}$$

As a consequence:

$$\lim_{t\to\infty} P_z^u\{\zeta_t \leq y\} = \frac{G(y)}{G(\infty)}, \quad y \in \mathbb{R}.$$

It follows from the properties of the function $\beta(\cdot)$ introduced in (4.5) that the limiting distributions of the processes (ζ_t) and (ξ_t) exist simultaneously. Consequently, the probabili distribution function $\frac{F(\cdot)}{F(\infty)}$, where

$$F(x) \triangleq G(\beta(x)) = \int_{-\infty}^{x} \frac{dy}{\beta'(y)} = \int_{-\infty}^{x} \exp\{2 \int_0^y a(z)dz\}dy, \tag{5.6}$$

is invariant for the Markov process $\{\Omega, \mathcal{F}_T, \mathcal{F}_t, \xi_t, P_x^u\}$. The ergodi properties of the latter can be read off from those of the (ζ_t) process:

$$\int_{-\infty}^{\infty} P_x^u\{\xi_t \leq y\}dF(x) = F(y), \text{ any } \quad 0 \leq t \leq T,\ y \in \mathbb{R}. \tag{5.3)'$$

$$\frac{1}{T}\int_0^T f(\xi_t)dt \xrightarrow[T\to\infty]{} \frac{1}{F(\infty)} \int_{-\infty}^{\infty} f(y)dF(y), \text{ a.s. } (P_x^u) \text{ and } L^1(E^u_x \tag{5.4)'$$

$$\lim_{t\to\infty} E_x^u f(\xi_t) = \frac{1}{F(\infty)} \int_{-\infty}^{\infty} f(y)dF(y), \tag{5.5)'$$

any Borel function $f(\cdot)$ such that:

$$\int_{-\infty}^{\infty} |f(y)|\,dF(y) < \infty, \text{ any } \quad x = \xi_0 \in \mathbb{R}.$$

Proposition 5.1: For any law $u \in \mathcal{U}$, $u_t(\xi) = a(\xi_t)$, the corre sponding solution process $\xi_t^u = \xi_t$ of the system equation (4.4) is a strongly Feller process, pocessing a unique invariant probability distribution $F^u(\cdot)/F^u(\infty)$, $F^u(x) = F(x)$ as in (5.6), for which (5.3)'-(5.5)'hold.

Proof. All that remains to be proven is the strong Feller property and the uniqueness of the invariant distribution, and it suffices to do both on the (ζ_t) process. The latter is indeed strongly Feller, since (5.1) implies a fortiori: $\sigma^2(z) \geq \sigma^2$, all $z \in \mathbb{R}$, some $\sigma^2 > 0$; see Wonham [10]. On the other hand (ζ_t) is recurre and positive, by (5.2). For such processes, Khas'minskii [7] proves the existence of a unique invariant distribution, Q.E.D.

Definition 5.2. For the constant α of (2.5), $0 < \alpha < 2$, let $\mathcal{U}_\alpha$ be the subclass of $\mathcal{U}$ consisting of those laws u, $u_t(\xi) = a(\xi_t)$ for which

$$(5.7) \qquad \int_{-\infty}^{\infty} e^{\alpha|x|}\,dF^u(x) = \int_{-\infty}^{\infty} e^{\alpha|x|} \exp\{2\int_0^x a(z)\,dz\}\,dx < \infty.$$

It is evident from (5.4)' and the assumption (2.5) that, for any $u \in \mathcal{U}_\alpha$:

$$J(u) = J(x,u) = \lim_{T\to\infty} \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t) + |u_t(\xi)|]dt$$

(5.8)

$$= \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} [\phi(y) + |a(y)|]dF^u(y),$$

any $x = \xi_0 \in \mathbb{R}$.

6. THE OPTIMAL LAW IN $\mathscr{L}_\alpha$

Introduce the function

(6.1)
$$c(p) \triangleq \min_{|u|\leq 1} (up+|u|) = 1 - |p|, \ |p| \geq 1$$
$$= 0, \ |p| < 1.$$

Our objective is to find a positive constant λ and a function $\nu(x)$, twice continuously differentiable on $\mathbb{R}$ and $O(e^{\alpha|x|})$ as $|x| \to \infty$, with $0 < \alpha < 2$ as in (2.5), satisfying the Dynamic Programming equation

(6.2)
$$\lambda = \frac{1}{2} \nu_{xx}(x) + c(\nu_x(x)) + \phi(x), \quad x \in \mathbb{R}.$$

We start with a preliminary result.

<u>Lemma 6.1</u>. Under the assumptions on the running cost function $\phi(\cdot)$ made in section 2, there exists a unique solution (λ,b) to the pair of equations

(6.3)
$$\lambda b - \int_0^b \phi(s)ds = \frac{1}{2}$$

$$\lambda = 2\int_0^\infty e^{-2s}\phi(b+s)ds. \tag{6.4}$$

Proof. It suffices to prove that equation $H(x) = 0$,

$$H(x) = 2x\int_0^\infty e^{-2s}\phi(x+s)ds - \int_0^x \phi(s)ds - \frac{1}{2}, \tag{6.5}$$

has a unique solution b on $\mathbb{R}^+$. Indeed, $H(0) = -1/2$ and

$$H'(x) = 2\int_0^\infty e^{-2s}[\phi(x+s) - \phi(x)]ds + 2x\int_0^\infty e^{-2s}\phi'(x+s)ds \geq \lambda\phi'(x), \quad x > 0.$$

Clearly, $H(x) \to \infty$ as $x \to \infty$, so there exists a unique number $b > 0$, such that $H(b) = 0$, Q.E.D.

The constants (λ, b) being as in the previous Lemma, consider the function $\nu(x)$ with $\nu(0) = 0$ and derivative given by

$$\begin{aligned} \nu_x(x) &= 2\lambda x - 2\int_0^x \phi(s)ds \qquad &; \quad 0 \leq x \leq b \\ &= 1 + \lambda[e^{2(x-b)} - 1] - 2\int_b^x e^{2(x-s)}\phi(s)ds; \qquad & x > b \\ &= -\nu_x(-x) \qquad &; \quad x < b. \end{aligned} \tag{6.6}$$

Proposition 6.2. The function $\nu(x)$ defined above is the unique (up to an additive constant) solution of (6.2) in $C^{(2)}(\mathbb{R})$, with λ determined along with the constant b through (6.3)-(6.4). $\nu(x)$ is also the smallest nonnegative function satisfying equation (6.2).

Proof. From (6.6), $\nu_x(b+) = 1$ while $\nu_x(b-) = 2\lambda b - 2\int_0^b \phi(s)ds$ by (6.3). Therefore $\nu_x(x)$ is continuous on $\mathbb{R}$. On the other han

$$
\begin{aligned}
\nu_{xx}(x) &= 2(\lambda-\phi(x)) &&; 0 \le x \le b \\
&= 2\left[\lambda e^{2(x-b)} - \phi(x) - 2\int_b^x e^{2(x-s)}\phi(s)ds\right]; && x > b \qquad (6.7)\\
&= \nu_{xx}(-x) &&; x < 0
\end{aligned}
$$

is clearly continuous on $\mathbb{R}$. From (6.4) and the fact that $\phi(\cdot)$ is strictly increasing on $\mathbb{R}^+$ one gets: $\nu_{xx}(x) \ge 2(\lambda-\phi(b)) > 0$, on $0 \le x \le b$, as well as

$$\nu_{xx}(x) = 2\left[2\int_x^\infty e^{-2(s-x)}\phi(s)ds - \phi(x)\right] > 0, \text{ on } x > b.$$

The function $\nu(x)$ is even and strictly convex, therefore minimal among nonnegative solutions of (6.2). By strict convexity, $0 < \nu_x(x) < 1$, on $0 < x < b$ and $\nu_x(x) > 1$, on $x > b$. It remains to verify (6.2), which in the present case becomes

$$
\begin{aligned}
\lambda &= \tfrac{1}{2}\nu_{xx}(x) + \phi(x) &&; |x| \le b \\
&= \tfrac{1}{2}\nu_{xx}(x) + 1 - \nu_x(x) + \phi(x); && x > b \qquad (6.2)'\\
&= \tfrac{1}{2}\nu_{xx}(x) + 1 + \nu_x(x) + \phi(x); && x < -b.
\end{aligned}
$$

(6.2)' is readily verified, by substitution. Uniqueness of $\nu_x(x)$ is a consequence of Lipschitz continuity of the function $c(p)$ defined in (6.1).

<u>Proposition 6.3</u>. Suppose that $\tilde{\lambda}, \tilde{\nu}(x)$ are a constant a $C^{(2)}(\mathbb{R})$ function, respectively, for which (6.2) is satisfied, and such that

(i) $0 < \tilde{\nu}_x(x) < 1, \ 0 < x < \tilde{b}$

(ii) $\tilde{\nu}_x(x) = 1,$

(iii) $\tilde{\nu}_x(x) > 1, \ x > \tilde{b},$

for some positive constant $\tilde{b}$.

Then the function $\tilde{\nu}(x)$ is necessarily strictly convex, therefore $\tilde{\nu}_x(x)$ is strictly increasing, $\tilde{b} \leq b$ and

$$\tilde{\lambda} \geq \lambda. \tag{6.8}$$

<u>Proof</u>. It is a straightforward exercise to verify that $\tilde{\nu}_x(x)$ will necessarily be of the form (6.6), with $(\tilde{\lambda}, \tilde{b})$ replacing (λ, b). A necessary and sufficient condition for continuity of $\tilde{\nu}_x(x)$ is then

$$\tilde{\lambda}\tilde{b} - \int_0^{\tilde{b}} \phi(s)ds = 1/2, \tag{$\widetilde{6.3}$}$$

while (iii) implies

$$\tilde{\lambda} > \frac{2\int_b^x e^{2(x-y)}\phi(y)dy}{e^{2(x-\tilde{b})}-1} = \frac{2\int_0^{x-\tilde{b}} e^{-2s}\phi(\tilde{b}+s)ds}{1-e^{-2(x-\tilde{b})}}, \text{ all } x > \tilde{b}. \tag{6.9}$$

A necessary and sufficient condition for (6.9) is (6.10) below:

$$\tilde{\lambda} \geq 2 \int_0^\infty e^{-2s}\phi(\tilde{b}+s)ds. \tag{6.10}$$

Indeed, if $\tilde{\lambda} < 2 \int_0^\infty e^{-2s}\phi(\tilde{b}+s)ds$ holds, then (6.9) is eventually false as $x \to \infty$. On the other hand, suppose that (6.10) is true; to prove (6.9) it suffices to prove

$$(1-e^{-2t}) \int_0^\infty \phi(\tilde{b}+s)e^{-2s}ds > \int_0^t \phi(\tilde{b}+s)e^{-2s}ds, \text{ all } t > 0 \tag{6.11}$$

where $t = x - \tilde{b}$. But (6.11) is equivalent to:

$$\int_t^\infty e^{-2s}[\phi(\tilde{b}+s) - \phi(\tilde{b}+s-t)]ds > 0, \quad \text{all} \quad t > 0,$$

which is obviously true since $\phi(\cdot)$ is strictly increasing.

Relations $\widetilde{(6.3)}$, (6.10) are therefore necessary and sufficient conditions for the feasibility of (i)-(iii). They imply that $H(\tilde{b}) \leq 0$, $H(\cdot)$ being the function introduced in (6.5). But $H(\cdot)$ is strictly increasing so $\tilde{b} \leq b$ and therefore $\tilde{\lambda} \geq \lambda$, from (6.3) and (6.10). Strict convexity of $\tilde{v}(x)$ is proven as in Proposition 6.2, Q.E.D.

Once the solution of the dynamic programming equation (6.2) corresponding to the smallest possible value of the constant λ has been constructed, we proceed to prove the main result of this section, namely the optimality in the class $\mathscr{C}_\alpha$ (Definition 5.2) of the law $u_t^*(\xi) = a^*(\xi_t)$, with

$$(6.12) \qquad a^*(x) = -\operatorname{sgn} x, \ |x| > b$$
$$= 0, \ |x| \le b$$

obtained through the minimization

$$(6.13) \quad a^*(x)\cdot\nu_x(x) + |a^*(x)| \equiv \min_{|u|\le 1} [u\cdot\nu_x(x) + |u|] = c(\nu_x(x)), \text{ all } x \in \mathbb{R}$$

Lemma 6.4. $\nu(x) = O(e^{\alpha|x|})$, as $|x| \to \infty$.

Proof. It is checked that for all x large enough

$$\nu_x(x) = 1 - \lambda + 2e^{2x}\int_x^\infty e^{-2y}\phi(y)dy \le 1 - \lambda + \frac{2c}{2-\alpha}e^{\alpha x},$$

some $c > 0$. The result follows readily.

Remark. Dr. Martin Day has noted that, for any other pair $(\tilde{\lambda},\tilde{b})$ as in Proposition 6.3, the functions $v_x(x), v(x)$ have a growth of the order $e^{2|x|}$, as $|x| \to \infty$.

Theorem 6.5. The law $u^* \in \mathcal{U}_\alpha$, defined through

$$u_t^*(\xi) = a^*(\xi_t), \text{ all } \xi \in C_{[0,T]}, \quad 0 \le t \le T,$$

with $a^*(\cdot)$ as in (6.12), is optimal in $\mathcal{U}_\alpha$. Furthermore:

$$J(u^*) = \lambda.$$

Proof. Consider any law $u \in \mathscr{U}_\alpha$ and the Markov process $\{\Omega, \mathscr{F}_T, \mathscr{F}_t, \xi_t^u, P_x^u\}$ -solution to the stochastic differential equation (4.4). An application of Itô's rule to the process $\nu(\xi_t^u)$, along with (6.13) and equation (6.2), yields

$$\begin{aligned}
\nu(\xi_t^u) - \nu(x) &= \int_0^t \left[\frac{1}{2}\nu_{xx}(\xi_s^u) + u_s(\xi^u)\nu_x(\xi_s^u)\right]ds + \int_0^t \nu_x(\xi_s^u)dw_s \\
&\geq \int_0^t \left[\frac{1}{2}\nu_{xx}(\xi_s^u) + c(\nu_x(\xi_s^u)) - |u_s(\xi^u)|\right]ds + \int_0^t \nu_x(\xi_s^u)dw_s \\
&\geq \lambda t - \int_0^t \left[\phi(\xi_s^u) + |u_s(\xi^u)|\right]ds + \int_0^t \nu_x(\xi_s^u)dw_s, \text{ a.s. } (P_x^u).
\end{aligned}$$

Taking expectations, and noting that

$$E_x^u \int_0^t \nu_x^2(\xi_s^u)ds \leq Const.e^{2\alpha(|x|+t)} E_x^u \int_0^t e^{2\alpha|w_s|} ds < \infty,$$

one gets:

$$\text{(6.14)} \qquad \frac{E_x^u \nu(\xi_t^u)}{t} - \frac{\nu(x)}{t} + \frac{1}{t} E_x^u \int_0^t \left[\phi(\xi_s^u) + |u_s(\xi^u)|\right]ds \geq \lambda, \quad \text{all } x.$$

From (5.5)', (5.7) and Lemma (6.4) one gets

$$\lim_{t\to\infty} E_x^u \nu(\xi_t^u) = \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} \nu(y)dF^u(y), \text{ any } x \in \mathbb{R}.$$

while taking (5.8) into account and letting $t \to \infty$ in (6.14):

$$J(u) \geq \lambda, \text{ any } u \in \mathscr{U}_\alpha.$$

On the other hand, (6.14) holds as an equality if $u = u^*$. Therefore

$$J(u^*) = \lambda .$$

The last two relations prove optimality of u^* in $\mathscr{S}_\alpha$. The density of $\mathscr{F}^{u^*}(\cdot)$ is given by

$$(6.15)\qquad \begin{aligned} p_*(y) &= (1+2b)^{-1}, & |y| \le b \\ &= (1+2b)^{-1}\exp[-2(|y|-b)], & |y| > b. \end{aligned}$$

7. OPTIMALITY OF THE LAW u^* in $\mathscr{U}$

In this section the performance of the law u^* of Theorem 6.5 is compared against the performance of any admissible nonanticipative control law u, and u^* is proven optimal in the class $\mathscr{U}$.

The method consists in considering the finite-horizon optimization problem: minimize

$$E_x^u \int_0^T [\phi(\xi_s) + |u_s(\xi)|]ds$$

subject to $d\xi_t = u_t(\xi)dt + dw_t$, $\xi_0 = x$ and $u \in \mathscr{U}$. Roughly speaking, the value function

$$V(x,\tau) = \inf_{u\in\mathscr{U}} E_x^u \int_{T-\tau}^T [\phi(\xi_s) + |u_x(\xi)|]ds; \ (x,\tau) \in \mathbb{R} \times [0,T]$$

solves the Cauchy problem

$$(7.1)\qquad V_\tau = \frac{1}{2} V_{xx} + c(V_x) + \phi(x); \quad (x,\tau) \in \mathbb{R} \times (0,T].$$

$$(7.2)\qquad V(x,0) = 0: \quad x \in \mathbb{R},$$

where $c(\cdot)$ is the function defined in (6.1).

For any law $u \in \mathcal{U}$, Itô's rule gives

$$E_x^u \int_0^T [\phi(\xi_s) + |u_s(\xi)|]ds \geq V(x,T),$$

and optimality of u^* would follow if it were proved that:

$$\lim_{T\to\infty} \frac{V(x,T)}{T} = \lambda, \quad \text{all} \quad x \in \mathbb{R}.$$

In the remaining of this section we justify the method and substantiate the above heuristics.

<u>Lemma 3.1</u>: <u>A priori bounds on the solution of the Bellman equation and its gradient</u>. Suppose that the Cauchy problem (7.1), (7.2) has a $C^{2,1}$ solution $V(x,\tau)$ on $\mathbb{R} \times (0,T]$, with $V(x,\tau)$ continuous on $\mathbb{R} \times [0,T]$. Then the following inequalities hold:

$$V(x,\tau) \leq \nu(x) + \lambda\tau, \text{ on } \mathbb{R} \times [0,T], \tag{7.3}$$

$$|V_x(x,\tau)| \leq \nu_x(|x|), \text{ on } \mathbb{R} \times [0,T]. \tag{7.4}$$

<u>Proof</u>. It is immediately verified that the function $M(x,\tau) = \nu(x) + \lambda\tau$ is a $C^{2,1}$ solution in $\mathbb{R} \times (0,T]$ of the Cauchy problem

$$M_\tau = \frac{1}{2} M_{xx} + c(M_x) + \phi(x); \quad \text{on } \mathbb{R} \times (0,T] \tag{7.5}$$

$$M(x,0) = \nu(x), \quad \text{on } \mathbb{R} \tag{7.6}$$

and that, if $\mathscr{L}$ is the parabolic operator

$$\mathscr{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{c(\nu_x)-c(V_x)}{\nu_x - V_x}\frac{\partial}{\partial x} - \frac{\partial}{\partial \tau} .$$

then

$$\mathscr{L}(M-N) = 0, \quad \text{in } \mathbb{R} \times (0,T]$$
$$M(x,0) - V(x,0) = \nu(x) \geq 0, \text{ on } \mathbb{R}.$$

By the maximum principle (see [3]) one obtains (7.3).

Now consider a sequence $\{c_n(p),\ n \in \mathbb{N}\}$ of smooth (piecewise C^2) approximations to the function $c(p)$, with $\ddot{c}_n(p) \leq 0$ a.e. on along with the functions $V^{(n)}(x,\tau)$, $M^{(n)}(x,\tau)$ satisfying (7.1), (7.2) and (7.5), (7.6) respectively, with $c(\cdot)$ replaced by $c_n(\cdot)$ Under such an approximating scheme, $V^{(n)}(x,\tau)$, $V_x^{(n)}(x,\tau)$, $V_{xx}^{(n)}(x,\tau)$ converge as $n \to \infty$ to $V(x,\tau)$, $V_x(x,\tau)$, $V_{xx}(x,\tau)$ respectively, uniformly on compact (x,τ) sets. Similarly for the function $M(x,\tau)$ and its approximations.

It is easily checked that if $\mathscr{L}_1$ is the parabolic operator

$$(7.7) \qquad \mathscr{L}_1 = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \dot{c}_n(M_x^{(n)})\frac{\partial}{\partial x} + \frac{\dot{c}_n(M_x^{(n)})-\dot{c}_n(V_x^{(n)})}{M_x^{(n)}-V_x^{(n)}}V_{xx}^{(n)} \cdot - \frac{\partial}{\partial \tau} ,$$

then:

$$\mathscr{L}_1(M_x^{(n)}-V_x^{(n)}) = 0, \quad \text{on } \mathbb{R}^+ \times (0,T]$$

$$M_x^{(n)}(x,0) - V_x^{(n)}(x,0) = \nu_x(x) \geq 0, \quad \text{on } \mathbb{R}^+.$$

It can be shown by yet another application of the maximum principle that $V_{xx}^{(n)}(x,\tau) \geq 0$ on $\mathbb{R} \times [0,T]$. Therefore, the potential term $\frac{\dot{c}_n(M_x^{(n)}) - \dot{c}_n(V_x^{(n)})}{M_x^{(n)} - V_x^{(n)}} V_{xx}^{(n)}$ is nonpositive on $\mathbb{R}^+ \times [0,T]$, so that the the strong maximum principle is applicable (see [3]) and gives $V_x^{(n)}(x,\tau) \leq M_x^{(n)}(x,\tau)$, or $V_x(x,\tau) \leq M_x(x,\tau) = \nu_x(x)$ on $\mathbb{R}^+ \times [0,T]$ in the limit as $n \to \infty$. (7.4) follows since $V_x(\cdot,\tau)$ is odd, Q.E.D.

Once the a priori bounds (7.3), (7.4) have been established, one can apply the method of Theorem VI 6.2 of Fleming and Rishel [2] to prove the following result:

Proposition 7.2. The Cauchy problem (7.1)-(7.2) has a unique $C^{2,1}$ solution $V(x,\tau)$ on $\mathbb{R} \times (0,T]$ that is continuous on $\mathbb{R} \times [0,T]$ and even in x.

By the approximation argument used in the proof of Lemma 7.1 (or directly; see [2], Exercise VI.9) it can be shown that $V_{xx}(x,\tau) \geq 0$ in $\mathbb{R} \times [0,T]$.

Consider the optimal process (η_t^τ) for the finite horizon problem, defined on the probability space $(\Omega, \mathcal{F}, P)$ as the (strong) solution of the stochastic differential equation

$$(7.8) \qquad d\eta_t^\tau = \dot{c}(V_x(\eta_t^\tau, \tau - t))dt + dw_t; \quad 0 \leq t \leq \tau$$

$$(7.9) \qquad \eta_0^\tau = x > 0$$

where $\dot{c}(p) = -\mathrm{sgn}p \cdot 1_{\{|p|>1\}} = a^*(p)$.

Lemma 7.3. For any $x > 0$, consider the stopping time

$$S = \inf\{t \leq \tau;\ \eta_t^\tau = 0\}$$
$$= \tau, \text{ if } \eta_t^\tau > 0, \text{ all } 0 \leq t \leq \tau.$$

Then

$$V_x(x,\tau) = E\int_0^S \dot{\phi}(\eta_t^\tau)dt. \tag{7.10}$$

Proof. The gradient V_x of the solution to the Cauchy problem (7.1)-(7.2) is not a $C^{2,1}$ function; it belongs, however, to the Sobolev space $W_p^{2,1}(D \times [0,T])$, for any $p > 1$ and any bounded subset $D \subseteq \mathbb{R}$, and satisfies in that space the equation $(V_x)_\tau = \frac{1}{2}(V_x)_{xx} + \dot{c}(V_x)(V_x)_x + \dot{\phi}(x)$ on $\mathbb{R} \times (0,T]$, derived from (7.1) by formal differentiation. For functions in the Sobolev space a generalized Itô formula holds (Zvonkin [12], Theorem 3) which, applied to $V_x(\eta_t^\tau, \tau - t)$ on $[0,S]$ along with (7.8) and the fact that $V_x(\eta_S^\tau, \tau - S) = 0$, a.s., yields (7.10), Q.E.D.

Consider now the "optimal process (ξ_t^*) for the stationary control problem":

$$d\xi_t^* = \dot{c}(v_x(\xi_t^*))dt + dw_t, \quad t \geq 0 \tag{7.11}$$

$$\xi_0^* = x, \tag{7.12}$$

defined on the same probability space $(\Omega, \mathcal{F}, P)$ and with the same initial condition as for (7.8), (7.9).

<u>Lemma 7.4</u>. $|\xi_t^*| \leq |\eta_t^\tau| \leq |x + w_t|$, $0 \leq t \leq \tau$; a.s.(P).

<u>Proof</u>. An easy consequence of the comparison theorem of Ikeda and Watanabe [6] and (7.4) of Lemma 7.1.

From (7.10) notice that, for any $\tau > 0$, $V_x(\cdot,\tau)$ increases to infinity as $x \to \infty$, since $\phi(\cdot)$ does. Therefore, for any $\tau \to 0$,

$$s(\tau) \overset{\Delta}{=} \max\{x > 0;\ V_x(x,\tau) = 1\} \tag{7.13}$$

is well-defined and finite.

<u>Lemma 7.5</u>. $s(\tau)$ is left continuous and decreasing on $\mathbb{R}^+$.

<u>Proof</u>. It can be checked that for the approximating functions introduced in the proof of Lemma 7.1: $\mathcal{L}_1(V_{x\tau}^{(n)}) = 0$ in $\mathbb{R} \times (0,T]$, $\mathcal{L}_1$ being the operator defined in (7.7), and $V_{x\tau}^{(n)}(x,0) = \phi(x) \geq 0$, on $\mathbb{R}^+$. By a maximum principle argument: $V_{x\tau}^{(n)}(x,\tau) \geq 0$ on $\mathbb{R}^+ \times [0,T]$, and therefore $V_x(x,\tau_2) \geq V_x(x,\tau_1)$, $0 \leq \tau_1 < \tau_2$, $x \geq 0$ in the limit as $n \to \infty$. This proves the monotonicity of $s(\cdot)$. Left continuity is an easy consequence of definition (7.13) and monotonic

<u>Lemma 7.6</u>. $\lim_{\tau\to\infty} V_x(x,\tau) = v_x(x)$, uniformly on compact x-sets.

Proof. Notice that

$$(\nu_x - V_x)_\tau = \frac{1}{2}(\nu_x - V_x)_{xx} + \dot{c}(\nu_x)(\nu_x - V_x)_x + V_{xx}(\dot{c}(\nu_x) - \dot{c}(V_x))$$

$$\leq \frac{1}{2}(\nu_x - V_x)_{xx} + \dot{c}(\nu_x)(\nu_x - V_x)_x,$$

on $\mathbb{R} \times (0,T]$, by convexity of V, monotonicity of $\dot{c}$ and (7.4). An application of the generalized Itô formula to $\nu_x(\xi_t^*) - V_x(\xi_t^*, \tau - t)$ gives:

$$0 \leq \nu_x(x) - V_x(x,\tau) \leq E\, \nu_x(\xi_R^*) = \int_{\{R=\tau\}} \nu_x(\xi_\tau^*)\,dP,$$

where:

$$R = \inf\{t \leq \tau : \xi_t^* = 0\}$$

$$= \tau, \text{ if } \xi_t^* > 0, \text{ all } 0 \leq t \leq \tau.$$

We note that: $E\, \nu_x^{1+\delta}(|\xi_\tau^*|) \xrightarrow[\tau\uparrow\infty]{} \int_{-\infty}^{\infty} \nu_x^{1+\delta}(|y|)p_*(y)dy < \infty$ as long as $0 < \delta < \frac{2}{\alpha} - 1$, by virtue of (5.5) and (6.15). So $\sup_{\tau>0} E\nu_x^{1+\delta}(|\xi_\tau^*|) < \infty$, which implies uniform integrability (and hence also absolute continuity with respect to measure P) of the family of random variables $\{\nu_x(|\xi_\tau^*|)\}_{\tau>0}$. On the other hand,

$$P(R = \tau) \leq P(x + w_t > 0, \text{ all } 0 \leq t \leq \tau) = 2\Phi(x\tau^{-1/2}) - 1 \to$$

as $\tau \to \infty$, uniformly on compact x-sets; see Gihman and Skorohod [4; §1]. The result follows by uniform absolute continuity.

Corollary. $s(\tau) \downarrow b$, as $\tau \uparrow \infty$.

Proposition 7.7. $\lim_{\tau\to\infty} \frac{V(x,\tau)}{\tau} = \lambda$, any $x \in \mathbb{R}$.

Proof. That $\limsup_{\tau\to\infty} \frac{V(x,\tau)}{\tau} \leq \lambda$, uniformly on compact x-sets, is a consequence of (7.3). To prove the opposite inequality note that, by virtue of Lemma 7.4,

$$V(x,\tau) = E\int_0^\tau [\phi(\eta_t^\tau) + 1_{\{|\eta_t^\tau|>s(\tau-t)\}}]dt$$

$$\geq E\int_0^\tau [\phi(\xi_t^*) + 1_{\{|\xi_t^*|>s(\tau-t)\}}]dt$$

and therefore, for any $x \in \mathbb{R}$,

$$\frac{V(x,\tau)}{\tau} > \frac{1}{\tau} E\int_0^\tau [\phi(\xi_t^*) + 1_{\{|\xi_t^*|>b\}}]dt -$$

$$(7.14) \qquad - \frac{1}{\tau}\int_0^\tau [F_{t,x}(s(\tau-t)) - F_{t,x}(b) - \{F_{t,x}(-s(\tau-t)) - F_{t,x}(-b)\}]dt$$

where

$$F_{t,x}(y) = P\{\xi_t^* \leq y \mid \xi_0^* = x\} \xrightarrow[t\to\infty]{} F^*(y) = \int_{-\infty}^y p_*(z)dz.$$

$F^*(\cdot)$ is the ergodic probability distribution correspondin to the optimal law u^* in $\mathscr{U}_\alpha$. Now

$$(7.15) \qquad \lim_{\tau\uparrow\infty} \frac{1}{\tau} \int_0^\tau [\, F_{t,x}(s(\tau-t)) - F_{t,x}(b) - \{\, F^*(s(\tau-t)) - F^*(b)\}]dt = 0.$$

Indeed, the integrand in (7.15) is dominated by $2 \sup_{y\in\mathbb{R}} |\, F_{t,x}(y) - F^*(y)|$, which tends to zero as $t \to \infty$, because F^* is (absolutely) continuous and $F_{t,x} \xrightarrow{c} F^*$ (see [1], p. 25 Ex. 8.1.13). Hence

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau (F_{t,x}(s(\tau-t)) - F_{t,x}(b))dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau F^*(s(\tau-t)) - F^*(b))dt = 0$$

since $\lim_{t\to\infty} F^*(s(t)) = F^*(b)$, by the Corollary to Lemma 7.6. By the same token, the entire second term on the right hand side of (7.14) converges to zero as $\tau \to \infty$, while the first term converges to λ. Therefore, for any $x \in \mathbb{R}$:

$$\liminf_{\tau\to\infty} \frac{V(x,\tau)}{\tau} \geq \lambda, \qquad \text{Q.E.D.}$$

We are now in a position to prove the main result of this section.

<u>Theorem 7.8</u>. The law u^* of Theorem 6.5 is optimal in the class $\mathscr{U}$ of admissible nonanticipative controls, i.e. for any $u \in \mathscr{U}$, $x \in \mathbb{R}$:

$$(7.16) \qquad J(x,u) = \liminf_{T\to\infty} \frac{1}{T} E_x^u \int_0^\tau [\phi(\xi_t^u) + |u_t(\xi)|]dt \geq \lambda = J(u^*).$$

Proof. Take any law $u \in \mathcal{U}$ along with the Girsanov solution process (ξ_t^u) satisfying $d\xi_t^u = u_t(\xi^u)dt + d\tilde{w}_t$, $\xi_0^u = x$ on $(\Omega, \mathcal{F}_T, P_x^u)$ as in Section 2, and apply Itô's rule to the process $V(\xi_t^u, T-t)$, $V(x,\tau)$ being the function of Proposition 7.2:

$$V(x,T) = V(\xi_0^u,T) - V(\xi_T^u,0) = -\int_0^T [u_t(\xi^u)V_x(\xi_t^u,T-t)$$

$$+ \frac{1}{2} V_{xx}(\xi_t^u,T-t) - V_\tau(\xi_t^u,T-t)]dt - \int_0^\tau V_x(\xi_t^u,T-t)d\tilde{w}_t.$$

Because $c(p) = \min\limits_{|u|\leq 1} (up+|u|)$, we get

$$(7.17) \quad V(x,T) \leq \int_0^T [\phi(\xi_t^u) + |u_t(\xi^u)|]dt - \int_0^T V_x(\xi_t^u,T-t)d\tilde{w}_t \text{ a.s. } (P_x^u),$$

$$\text{any } x \in \mathbb{R},\ T > 0.$$

The expectation of the stochastic integral on the right hand side of (7.17) is zero, because

$$E_x^u \int_0^T V_x^2(\xi_t^u,T-t)dt \leq E_x^u \int_0^T V_x^2(\xi_t^u)dt \leq \text{const. } e^{2\alpha(|x|+T)} \int_0^T E(e^{2\alpha|w_t|})dt < \infty$$

by virtue of (7.4), and it follows from (7.17) by taking expectations that

$$\frac{V(x,T)}{T} \leq \frac{1}{T} E_x^u \int_0^T [\phi(\xi_t^u) + |u_t(\xi^u)|]dt; \text{ any } x \in \mathbb{R},\ T > 0.$$

(7.16) is obtained by a passage to the limit as $T \to \infty$ and taking into account the assertion of Proposition 7.7.

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